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Two-loop background field calculations for gauge theories with scalar fields

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Abstract. Recent calculations of the ultraviolet divergences of two-loop vacuum graphs in the presence of an arbitrary background field for a pure gauge theory are extended to arbitrary couplings to a scalar field on flat space.

1. Introduction

In a recent paper by Jack and Osborn (1982) the divergent behaviour of two-loop vacuum graphs in the presence of an arbitrary background field on flat space was analysed. Dimensional regularisation was employed and the pole terms in $\varepsilon = 4-d$ were determined in terms of the short distance behaviour of the exact propagators obtained using heat kernel techniques. The reader is referred to this paper, henceforth denoted by I, for basic details of the method and notation, and a comprehensive list of references. The procedure was there applied to the calculation of the divergences of the two-loop vacuum functional for a pure non-abelian gauge theory. In the present paper we extend this calculation to the case of a general gauge theory containing both vector and scalar fields. We reproduce the results of van Damme (1982) who adopted the method of 't Hooft (1973) (which at least to one-loop level is formally similar). In § 2 we explain the general formalism appropriate to all orders in the perturbation expansion, in § 3 we perform the one-loop renormalisation and in § 4 we calculate the two-loop counterterms. Some useful results pertaining to the heat kernels for the fluctuation operators involved are presented in appendix 1, while appendix 2 contains some identities derived from the gauge invariance of the scalar potential.

2. The general formalism

We consider a semi-simple group G and a real scalar field φ, regarded as a column vector with components φ_i, transforming under a representation of G with generators T_a, where

$$[T_a, T_b] = c_{abc} T_c, \qquad (T_a)^{\mathrm{T}} = -T_a, \qquad \mathrm{Tr}(T_a T_b) = -R_{ab}, \tag{2.1}$$

with c_{abc} totally antisymmetric. The appropriate covariant derivative acting on φ is then

$$D_\mu = \partial_\mu + A^\varphi_\mu, \qquad A^\varphi_\mu = A_{\mu,a} T_a, \tag{2.2}$$

where the gauge field belongs to the adjoint representation of G, with components $A_{\mu,a}$, and the corresponding generators $t_a^{\rm ad}$ in this basis satisfy

$$(t_a^{\rm ad})_{bc} = -c_{abc}, \qquad [t_a^{\rm ad}, t_b^{\rm ad}] = c_{abc} t_c^{\rm ad}, \qquad \mathrm{Tr}(t_a^{\rm ad} t_b^{\rm ad}) = -C_{ab}. \tag{2.3}$$

The covariant derivative in the adjoint representation is

$$\mathscr{D}_\mu = \partial_\mu + A_\mu^{\rm ad}, \qquad A_\mu^{\rm ad} = A_{\mu,a} t_a^{\rm ad}, \tag{2.4}$$

and the field strength then has components

$$F_{\mu\nu,a} = \partial_\mu A_{\nu,a} - \partial_\nu A_{\mu,a} + c_{abc} A_{\mu,b} A_{\nu,c}. \tag{2.5}$$

For X, Y in the adjoint representation the group invariant scalar product can be conveniently written

$$(XY) = X_a Y_a. \tag{2.6}$$

In perturbative calculations, quantities are expanded systematically in $\hbar$ or the number of loops, as

$$Q = \sum_n Q^{(n)} \hbar^n. \tag{2.7}$$

The renormalised action which appears in the functional integral is, in d dimensions,

$$S[A, \varphi] = \int \mathrm{d}^d x \left(\frac{1}{4g^2} (F_{\mu\nu} Z_A F_{\mu\nu}) + \tfrac{1}{2} (D_\mu \varphi)^{\mathrm{T}} Z_\varphi D_\mu \varphi + V(\varphi) \right) \tag{2.8}$$

where Z_A, Z_φ are group invariant matrices acting on the adjoint representation and on φ respectively. Together they and also V are regarded as expanded as in (2.7) with

$$Z^{(0)} = 1 \qquad \text{and} \qquad V^{(0)} = U(\varphi). \tag{2.9}$$

The quantum fields $A_\mu^{\rm q}$, $\varphi^{\rm q}$ are expanded about arbitrary background fields A_μ, φ as

$$A_\mu^{\rm q} = A_\mu + g\hbar^{1/2} Z_v^{1/2} v_\mu, \qquad \varphi^{\rm q} = \varphi + \hbar^{1/2} Z_f^{1/2} f, \tag{2.10}$$

where the fluctuations v_μ, f are integrated over in the functional integral and Z_v, Z_f depend on the (essentially arbitrary) overall scale of v_μ, f which will be specified later.

It is natural in the present background field calculation to combine the vector and scalar fluctuations v_μ and f into a single vector F,

$$F = \begin{pmatrix} v_{\mu,a} \\ f \end{pmatrix}, \tag{2.11}$$

with the scalar product (for flat space)

$$(F', F) = \int \mathrm{d}^d x ((v'_\mu v_\mu) + f'^{\mathrm{T}} f). \tag{2.12}$$

For a scalar field ω in the adjoint representation of G, we also define an associated gauge mode given by

$$\tilde{\mathscr{D}} \omega = \begin{pmatrix} (\mathscr{D}_\mu \omega)_a \\ -g \omega^\varphi \varphi \end{pmatrix} \tag{2.13}$$

where $\mathscr{D}_\mu$ is defined in (2.4). The corresponding scalar product for the scalar fields is

$$(\omega', \omega) = \int \mathrm{d}^d x \, (\omega' \omega) \tag{2.14}$$

so then $\tilde{\mathscr{D}}$ has an adjoint operator $\tilde{\mathscr{D}}^*$ satisfying

$$(\tilde{\mathscr{D}}\omega, F) = (\omega, \tilde{\mathscr{D}}^* F) \tag{2.15}$$

which requires

$$(\tilde{\mathscr{D}}^* F)_a = -(\mathscr{D} \cdot v)_a + g\varphi^{\mathrm{T}} T_a f. \tag{2.16}$$

The vacuum functional $W[A, \varphi, K]$ for a source K coupled just to the quantum fluctuations is then given by the functional integral

$$\mathrm{e}^W = \int \mathrm{d}[v, f] \Gamma \exp\left(-\frac{1}{\hbar} S(A^{\mathrm{q}}, \varphi^{\mathrm{q}}) - \frac{1}{\hbar} S_{\mathrm{gf}} + (K, F)\right) \tag{2.17}$$

where S_{gf} is an appropriate gauge-fixing term for the fluctuations v_μ, f and Γ is the corresponding Faddeev–Popov determinant. Background field methods enjoy particular simplicity when S_{gf} is chosen so that W is invariant under gauge transformations on the fields A_μ, φ. For $K = 0$, W is independent of the precise prescription for gauge fixing given by the choice of S_{gf}. In this case, if tadpole type diagrams are to give zero contributions, A_μ, φ are constrained to each order of the loop expansion with, to lowest order, $A_\mu^{(0)}$, $\varphi^{(0)}$ classical solutions for which $S^{(0)}$ is stationary. For arbitrary A_μ, φ, it is, on the other hand, possible to choose K so that W gets contributions only from one-particle-irreducible graphs. To each order $Z_{A,\varphi}^{(n)}$, $Z_{v,f}^{(n-1)}$ and $V^{(n)}$ are determined so as to ensure that $W^{(n)}[A, \varphi, K]$ is finite. We adopt the minimal subtraction scheme whereby these quantities are uniquely determined by containing just poles in ε.

The natural gauge-fixing term is expressed in terms of F by

$$S_{\mathrm{gf}} = \tfrac{1}{2}(\tilde{\mathscr{D}}^* F, \tilde{\mathscr{D}}^* F) \tag{2.18}$$

so that from (2.12) and (2.15)

$$S_{\mathrm{gf}} = \tfrac{1}{2}(F, \tilde{\mathscr{D}}\tilde{\mathscr{D}}^* F). \tag{2.19}$$

An infinitesimal gauge transformation on A_μ^{q}, φ^{q} gives rise to the fluctuation

$$F^{\mathrm{g}} = Z^{-1/2} \tilde{\mathscr{D}}^{\mathrm{q}} \omega, \qquad Z = \begin{pmatrix} Z_v & 0 \\ 0 & Z_f \end{pmatrix}, \qquad \tilde{\mathscr{D}}^{\mathrm{q}} \omega = \begin{pmatrix} (\mathscr{D}^{\mathrm{q}} \omega)_a \\ -g\omega^{\varphi} \varphi^{\mathrm{q}} \end{pmatrix} \tag{2.20}$$

(coinciding with (2.13) for $\hbar = 0$). From (2.18) and (2.20) the standard argument leads to a ghost determinant

$$\Gamma = \det(\tilde{\mathscr{D}}^* Z^{-1/2} \tilde{\mathscr{D}}^{\mathrm{q}}). \tag{2.21}$$

The choice (2.18), in conjunction with (2.21), then fixes the scale of F.

The action has an expansion expressible in terms of $F_{\mathrm{R}} = Z^{1/2} F$ as follows:

$$S[A^{\mathrm{q}}, \varphi^{\mathrm{q}}] = S[A, \varphi] + (J, F_{\mathrm{R}}) + \tfrac{1}{2}(F_{\mathrm{R}}, M F_{\mathrm{R}}) + S_{\mathrm{I}}(F_{\mathrm{R}}) \tag{2.22}$$

where

$$J = \begin{pmatrix} (1/g)(Z_A \mathscr{D}_\nu F_{\mu\nu})_a - g\varphi^{\mathrm{T}} T_a Z_\varphi D_\mu \varphi \\ -Z_\varphi D^2 \varphi + V'(\varphi) \end{pmatrix}, \tag{2.23}$$

$$M=\begin{pmatrix}(Z_A\Delta_{\mu\nu})_{ab}+g^2\delta_{\mu\nu}(T_a\varphi)^{\mathrm{T}}Z_\varphi T_b\varphi & g(-\varphi^{\mathrm{T}}T_aZ_\varphi D_\mu+(D_\mu\varphi)^{\mathrm{T}}Z_\varphi T_a)\\ g(-2T_bZ_\varphi D_\nu\varphi-T_cZ_\varphi\varphi(\mathscr{D}_\nu)_{cb}) & -Z_\varphi D^2+V''(\varphi)\end{pmatrix}, \tag{2.24}$$

introducing the notation

$$\Delta_{\mu\nu}=-\delta_{\mu\nu}\mathscr{D}^2+\mathscr{D}_\mu\mathscr{D}_\nu-2F^{\mathrm{ad}}_{\mu\nu}, \tag{2.25a}$$

$$V'_i(\varphi)=\partial V(\varphi)/\partial\varphi_i, \qquad V''_{ij}(\varphi)=\partial^2V(\varphi)/\partial\varphi_i\partial\varphi_j. \tag{2.25b}$$

Since

$$\tilde{\mathscr{D}}\tilde{\mathscr{D}}^*=\begin{pmatrix}-(\mathscr{D}_\mu\mathscr{D}_\nu)_{ab} & g((D_\mu\varphi)^{\mathrm{T}}T_a+\varphi^{\mathrm{T}}T_\alpha D_\mu)\\ gT_c\varphi(\mathscr{D}_\nu)_{cb} & Q(\varphi)\end{pmatrix} \tag{2.26}$$

where

$$Q(\varphi)=g^2T_a\varphi(T_a\varphi)^{\mathrm{T}} \tag{2.27}$$

when we combine (2.19) and (2.24) we obtain the effective zeroth-order operator on the quadratic fluctuations in the form

$$\Delta_F=M^{(0)}+\tilde{\mathscr{D}}\tilde{\mathscr{D}}^*=\begin{pmatrix}(\Delta^1_{\mu\nu})_{ab}+\delta_{\mu\nu}P_{ab}(\varphi) & 2g(D_\mu\varphi)^{\mathrm{T}}T_a\\ -2gT_bD_\nu\varphi & -D^2+U''(\varphi)+Q(\varphi)\end{pmatrix} \tag{2.28}$$

with

$$\Delta^1_{\mu\nu}=\Delta_{\mu\nu}-\mathscr{D}_\mu\mathscr{D}_\nu, \tag{2.29a}$$

$$P_{ab}=g^2(T_a\varphi)^{\mathrm{T}}T_b\varphi. \tag{2.29b}$$

The Green functions $\mathscr{G}^F$ and G^g for Δ_F and $\tilde{\mathscr{D}}^*\mathscr{D}$ yield the propagators for the vectors, scalars and Faddeev–Popov ghosts, represented in diagrams by wavy, straight and broken lines respectively. We write $\mathscr{G}^F$ as a matrix according to the decomposition (2.11)

$$\mathscr{G}^F(x,y)=\begin{pmatrix}G^F_{\mu\nu,ab}(x,y) & \bar{G}^F_{\mu,aj}(x,y)\\ G^F_{\nu,ib}(x,y) & G^F_{ij}(x,y)\end{pmatrix}. \tag{2.30}$$

Since Δ_F is symmetric this has the symmetry property $\mathscr{G}^F(x,y)=[\mathscr{G}^F(y,x)]^{\mathrm{T}}$, so that in particular

$$G^F_{\mu,ia}(x,y)=\bar{G}^F_{\mu,ai}(y,x). \tag{2.31}$$

The ghost determinant in (2.21) may be rewritten

$$\begin{aligned}\Gamma&=\det(\tilde{\mathscr{D}}^*\tilde{\mathscr{D}})\exp\operatorname{Tr}\ln(1+G^gE),\\ E_{ab}&=(\tilde{\mathscr{D}}^*(Z^{-1/2}-1)\tilde{\mathscr{D}})_{ab}+g(\mathscr{D}\cdot v)^{\mathrm{ad}}_{ab}+g^2\varphi^{\mathrm{T}}T_aT_bf.\end{aligned} \tag{2.32}$$

In perturbative calculations, when W is given by the sum of connected vacuum graphs, in our formalism it is necessary to include Feynman diagrams with mixed vector-scalar lines, in addition to the usual vector and scalar lines, represented by the elements of $\mathscr{G}^F$ in (2.30), and also ghost lines given by G^g resulting from the expansion of $\ln(1+G^gE)$ in (2.32).

3. One-loop renormalisation

In this section we briefly recapitulate the one-loop perturbative evaluation of the vacuum functional and derive Z_f and Z_v to one-loop order. We shall set $\hbar = 1$.

The zero-loop vacuum functional is simply the classical action,

$$W^{(0)} = -S^{(0)}[A, \varphi]. \tag{3.1}$$

For the one-loop contribution we take $K = J^{(0)}$ and the result is then

$$W^{(1)}[A, \varphi] = -S^{(1)}[A, \varphi] - \tfrac{1}{2}\ln(\det \Delta_F/\det \Delta_{F_0}) + \ln(\det \tilde{\mathscr{D}}^*\tilde{\mathscr{D}}/\det (\tilde{\mathscr{D}}^*\tilde{\mathscr{D}})_0) \tag{3.2}$$

where $\Delta_0 = \lim_{|x|\to\infty} \Delta$. We assume that the ratios of determinants in (3.2) are infrared finite. For evaluating the ultraviolet divergent behaviour of the determinant of an elliptic differential operator on flat space of the form

$$\Delta_x = -1D^2 + Y(x), \qquad D_\mu = \partial_\mu + X_\mu(x), \tag{3.3}$$

where X_μ and Y are general matrices, the heat kernel method expounded in I leads to the following prescription:

$$(-\ln(\det \Delta/\det \Delta_0))^{\text{pole}} = (16\pi^2\varepsilon)^{-1} \int d^dx \, \mathrm{Tr}(c^\Delta - c^{\Delta_0}), \tag{3.4a}$$

$$c^\Delta = (\tfrac{1}{6}G_{\mu\nu}G_{\mu\nu} + Y^2), \tag{3.4b}$$

$$G_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu + [X_\mu, X_\nu]. \tag{3.4c}$$

Of course this gives rise to precisely the same counter-Lagrangian as obtained by 't Hooft (1973).

For the two operators with which we are concerned here, Δ_F and $\tilde{\mathscr{D}}^*\mathscr{D}$, we have

$$X^F_\mu = \begin{pmatrix} A^{\text{ad}}_\mu & 0 \\ 0 & A^\varphi_\mu \end{pmatrix}, \qquad X^g_\mu = A^{\text{ad}}_\mu, \tag{3.5}$$

$$Y^F = \begin{pmatrix} (-2F^{\text{ad}}_{\mu\nu} + \delta_{\mu\nu}P(\varphi))_{ab} & 2g(D_\mu\varphi)^{\mathrm{T}}T_a \\ -2gT_bD_\nu\varphi & U''(\varphi) + Q(\varphi) \end{pmatrix}, \qquad Y^g = P(\varphi). \tag{3.6}$$

Substituting these quantities in turn into (3.4), and using (2.1), (2.3) and (A2.3), we find for the pole contribution in (3.2) from the one-loop vacuum graphs

$$(16\pi^2\varepsilon)^{-1} \int d^dx \{\tfrac{11}{6}C_{ab}F^a_{\mu\nu}F^b_{\mu\nu} - \tfrac{1}{12}R_{ab}F^a_{\mu\nu}F^b_{\mu\nu} + 4g^2\varphi^{\mathrm{T}}T^2D^2\varphi + \tfrac{3}{2}\,\mathrm{Tr}[P(\varphi)^2] + \tfrac{1}{2}\,\mathrm{Tr}[U''(\varphi)^2] - g^2U'(\varphi)^{\mathrm{T}}T^2\varphi\} \tag{3.7}$$

which may be cancelled by taking in $S^{(1)}[A, \varphi]$

$$Z^{(1)}_A = (1/\varepsilon)(g^2/16\pi^2)(\tfrac{22}{3}C - \tfrac{1}{3}R), \tag{3.8a}$$

$$Z^{(1)}_\varphi = -(1/\varepsilon)(g^2/16\pi^2)8T^2 \tag{3.8b}$$

$$V^{(1)}(\varphi) = (1/\varepsilon)(1/16\pi^2)\{\tfrac{1}{2}\mathrm{Tr}[U''(\varphi)^2] - g^2U'(\varphi)^{\mathrm{T}}T^2\varphi + \tfrac{3}{2}\mathrm{Tr}[P(\varphi)^2]\}. \tag{3.8c}$$

Manifestly only if $U(\varphi)$ is a general quartic gauge-invariant polynomial in φ can $V^{(1)}(\varphi)$ be absorbed in a redefinition of coupling constant in $V(\varphi)$. The two-loop calculations in the next section also determine the one-loop contributions in Z_n, Z_f.

These can be determined more simply and independently by requiring that

$$\langle F(x)\rangle = \int \mathrm{d}^d y\, \mathscr{G}^F(x,y)L(y), \qquad L = \begin{pmatrix} t_{\mu,a} \\ h_i \end{pmatrix}, \tag{3.9}$$

is finite, where L represents the F to vacuum amplitude. Our choice of $K = J^{(0)}$ ensures that $L^{(0)} = 0$ and $L^{(1)}$ is given by the one-loop graphs of figures 1, 2. To calculate these we need $S_{\mathrm{I}}^{(0)}$:

$$S_{\mathrm{I}}^{(0)} = \int \mathrm{d}^d x\{g(\mathscr{D}_\alpha v_\beta[v_\alpha, v_\beta]) + \tfrac{1}{4}g^2([v_\alpha, v_\beta][v_\alpha, v_\beta]) + g(v_\mu^\varphi f)^{\mathrm{T}}(D_\mu f + g v_\mu^\varphi \varphi + \tfrac{1}{2} g v_\mu^\varphi f) + (1/3!)U'''_{ijk} f_i f_j f_k + (1/4!)U''''_{ijkl} f_i f_j f_k f_l\}. \tag{3.10}$$

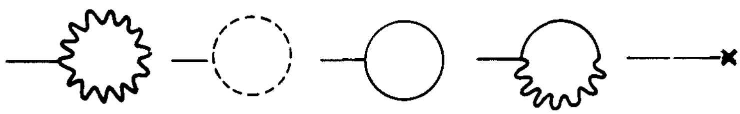

Figure 1. Contributions to $h_i^{(1)}$.

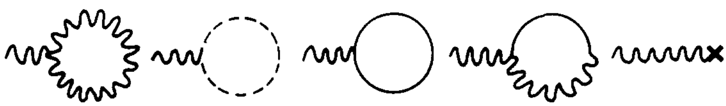

Figure 2. Contributions to $t_{\mu,a}^{(1)}$.

We then obtain

$$h_i^{(1)} = g^2(T_a T_b \varphi)_i (G^F_{\alpha\alpha,ab} - G^g_{ab})_{\mathrm{diag}} + g[T_a(2D_\mu G^F_\mu + G^F_\mu \overleftarrow{\mathscr{D}}_\mu)_{a_{\mathrm{diag}}}]_i - \tfrac{1}{2}U'''_{ijk} G^F_{jk_{\mathrm{diag}}} - V_i^{(1)\prime}(\varphi) + [(Z_f^{1/2} Z_\varphi)^{(1)} D^2\varphi]_i - [(Z_f^{1/2})^{(1)} T^2 U']_i, \tag{3.11}$$

$$t_{\mu,a}^{(1)} = g\,\mathrm{Tr}(t_a^{\mathrm{ad}}\{\mathscr{D}_\alpha G^F_{\alpha\mu} + \mathscr{D}_\mu G^F_{\alpha\alpha} - 2\mathscr{D}_\alpha G^F_{\mu\alpha} - \mathscr{D}_\mu G^g\}_{\mathrm{diag}}) + g\,\mathrm{Tr}(T_a D_\mu G^F_{\mathrm{diag}}) + g^2 \bar{G}^F_{\mu,b_{\mathrm{diag}}}\{T_a, T_b\}\varphi - (1/g)[(Z_v^{1/2} Z_A)^{(1)} \mathscr{D}_\nu F_{\mu\nu}]_a + g\varphi^{\mathrm{T}}((Z_v^{1/2} T)_a Z_\varphi)^{(1)} D_\mu \varphi. \tag{3.12}$$

In (3.10) and (3.11) we have used the notation that for a quantity $H(x, y)$, such as formed by the Green functions with various covariant derivatives acting on them, $H_{\mathrm{diag}}(x) = H(x, x)$. With dimensional regularisation $H_{\mathrm{diag}}(x)$ is well defined albeit containing, in general, poles in ε.

Using (A1.3a, b) and also (A2.8), we find that the finiteness of $L^{(1)}$ is ensured by taking

$$(Z_f^{1/2})^{(1)} = (1/\varepsilon)(2g^2/16\pi^2)T^2, \tag{3.13}$$

$$(Z_v^{1/2})^{(1)} = -(1/\varepsilon)(2g^2/16\pi^2)C. \tag{3.14}$$

4. The two-loop calculation

In this section we calculate the divergent part of the two-loop vacuum functional, which arises from the one-particle-irreducible diagrams shown in figure 3. The analysis of the divergences of the two-loop vacuum graphs depends on the detailed knowledge

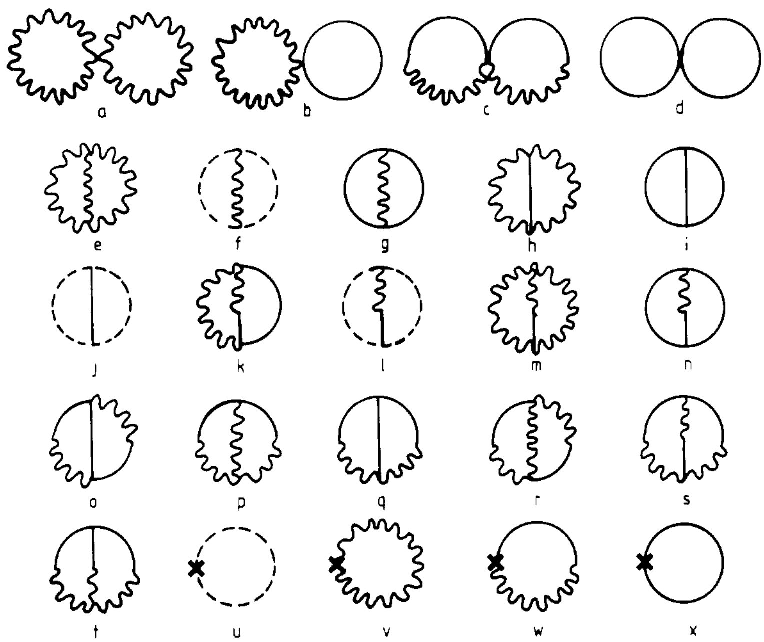

Figure 3. Two-loop 1PI graphs.

of the short distance behaviour of the propagators in the presence of the background fields in d dimensions described in I, briefly re-presented here in appendix 1. In the following discussion we make heavy use of the symmetry (2.31) to simplify the expressions involved.

There are two main classes of two-loop one-particle-irreducible vacuum diagrams. The first class are those in figures 3(a, b, c, d) which arise from a single quartic interaction, as contained in $S_I^{(0)}$, equation (3.10), and involve, for the two closed lines, propagators without derivatives, with coincident space arguments. For the four-gluon vertex the full expression was given in I, while for the additional diagrams resulting from the two-scalar, two-gluon vertex 3(b, c) and four-scalar vertex 3(d) the complete expressions, suppressing spatial arguments, are

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]_b = \tfrac{1}{2}g^2 \int \mathrm{d}^d x [\mathrm{Tr}(T_a G^F T_b) G^F_{\alpha\alpha,ab}], \tag{4.1}$$

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]_c = \tfrac{1}{2}g^2 \int \mathrm{d}^d x [\bar{G}^F_{\mu,a} T_a T_b G^F_{\mu,b} - \tfrac{1}{2}((T_a G^F_{\mu,a})^{\mathrm{T}} T_b G^F_{\mu,b} + (T_a G^F_{\mu,b})^{\mathrm{T}} T_b G_{\mu,a})], \tag{4.2}$$

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]_d = -\tfrac{1}{8} \int \mathrm{d}^d x [U''''_{ijkl} G^F_{ij} G^F_{kl}]. \tag{4.3}$$

The pole parts, easily obtained from (A1.3a), are

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]^{\mathrm{pole}}_a$$
$$= -\frac{6g^2}{(16\pi^2\varepsilon)^2} \int \mathrm{d}^d x [(F_{\mu\nu} C^2 F_{\mu\nu}) - 2(1-\tfrac{7}{12}\varepsilon)\,\mathrm{Tr}(t^{\mathrm{ad}}_a P t^{\mathrm{ad}}_a P)]$$
$$-\frac{3g^2}{16\pi^2\varepsilon} \int \mathrm{d}^d x [\mathrm{Tr}(C F^{\mathrm{ad}}_{\alpha\beta} H^F_{\beta\alpha_{\mathrm{diag}}}) + \mathrm{Tr}(t^{\mathrm{ad}}_a P t^{\mathrm{ad}}_a H^F_{\alpha\alpha_{\mathrm{diag}}})], \tag{4.4}$$

$$W^{(2)}_{1\text{PI}}[A,\varphi]^{\text{pole}}_b$$

$$=\frac{2dg^2}{(16\pi^2\varepsilon)^2}\int \mathrm{d}^dx[(-g^2P_{ab}\varphi T_aT_bT^2\varphi+\mathrm{Tr}(CP^2)+\mathrm{Tr}(t_a^{\mathrm{ad}}Pt_a^{\mathrm{ad}}P)$$

$$+P_{ab}\,\mathrm{Tr}(U''T_aT_b)]+\frac{g^2}{16\pi^2\varepsilon}\int \mathrm{d}^dx[-4P_{ab}\,\mathrm{Tr}(T_aT_bH^F_{\mathrm{diag}})$$

$$+g^2\varphi^{\mathrm{T}}T_aT_bT^2\varphi H^F_{\alpha\alpha,ab_{\mathrm{diag}}}-\mathrm{Tr}(t_a^{\mathrm{ad}}Pt_a^{\mathrm{ad}}H^F_{\alpha\alpha_{\mathrm{diag}}})-\mathrm{Tr}(CPH^F_{\alpha\alpha_{\mathrm{diag}}})], \qquad (4.5)$$

$$W^{(2)}_{1\text{PI}}[A,\varphi]^{\text{pole}}_c=\frac{8g^4}{(16\pi^2\varepsilon)^2}\int \mathrm{d}^dx[2\varphi^{\mathrm{T}}T^2T^2D^2\varphi+\tfrac{1}{2}C_{ab}\varphi^{\mathrm{T}}T_aT_bD^2\varphi]$$

$$-\frac{g^3}{8\pi^2\varepsilon}\int \mathrm{d}^dx[4(D_\mu\varphi)^{\mathrm{T}}T^2T_aH^F_{\mu,a_{\mathrm{diag}}}+C_{ab}(D_\mu\varphi)^{\mathrm{T}}T_aH^F_{\mu,b_{\mathrm{diag}}}], \qquad (4.6)$$

$$W^{(2)}_{1\text{PI}}[A,\varphi]^{\text{pole}}_d$$

$$=\frac{1}{2}\frac{1}{(16\pi^2\varepsilon)^2}\int \mathrm{d}^dx[-U''_{ij}U''_{kl}U''''_{ijkl}-2g^2U''_{kl}U''''_{ijkl}(T_a\varphi)_i(T_a\varphi)_j$$

$$+g^4U'_i(T^2T^2\varphi)_i+g^4\varphi^{\mathrm{T}}T^2U''T^2\varphi-2g^4\varphi^{\mathrm{T}}T_aT_bU''T_bT_a\varphi$$

$$+2g^4U'''_{ijk}(T_a\varphi)_i(T_a\varphi)_j(T^2\varphi)_k] \qquad (4.7)$$

$$+\frac{1}{2}\frac{1}{16\pi^2\varepsilon}\int \mathrm{d}^dx[U''_{ij}+g^2(T_a\varphi)_i(T_a\varphi)_j]U''''_{ijkl}H^F_{kl_{\mathrm{diag}}},$$

after using (A2.9) to simplify the expression for 3(d).

The remaining class of 1PI graphs involves two trilinear interactions from $S_{\mathrm{I}}^{(0)}$ or the ghost interaction, E in (2.32), linked by a product of three propagators $G(x,y)$. The interaction may have dimension four with derivatives, or dimension three. In the general expressions corresponding to figures 3(e–t), spatial arguments are again suppressed; in this case the forms have been arranged so that any functions or derivatives on the right of one of the G's implicitly depend on y, on the left on x. The pole contributions can be found using the methods of I for such products of Green functions. It is useful here to note that the off-diagonal components $G^F_{\mu,ai}$, $G^F_{\nu,jb}$ in (2.30) corresponding to mixed vector-scalar lines do not have the leading short-distance singularity as $x\to y$. These vacuum graphs produce both local pole terms arising from the leading short-distance singular behaviour of all three propagators, and non-local terms from the short distance behaviour of two propagators in conjunction with the part of the third regular in the short distance limit. (From (A1.1) there is a well defined decomposition into regular and singular parts.) Products of propagators where the leading short-distance singularity is $(x-y)^{-4-N}$ (in four dimensions), $N\geqslant 0$, with dimensional regularisation produce poles in ε multiplied by δ functions of $x-y$, with up to N derivatives. The resulting pole terms for $W^{(2)}$ then involve single x integrals.

Except for the cases where the full expression was given in I, we give below the complete amplitude for each diagram and then the resulting pole terms obtained using the analysis of I, in conjunction with (A1.2). The details are straightforward although sometimes a little tedious.

The most singular graphs are those with two dimension-four vertices shown in figures 3(e,f,g), for which N, mentioned above, takes the value 4. For 3(e,f), the full contributions were given in I and now give rise to pole terms

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]^{\mathrm{pole}}_e$$

$$= g^2(16\pi^2\varepsilon)^{-2}\int \mathrm{d}^dx[(6-\tfrac{1}{4}\varepsilon)(F_{\mu\nu}C^2F_{\mu\nu})+\tfrac{9}{2}(1-\tfrac{1}{3}\varepsilon)\,\mathrm{Tr}(CP^2)$$

$$+\tfrac{9}{2}g^2(1-\tfrac{1}{12}\varepsilon)C_{ab}\varphi^{\mathrm{T}}T_aT_bD^2\varphi-\tfrac{3}{2}(12-\varepsilon)\,\mathrm{Tr}(t_a^{\mathrm{ad}}Pt_a^{\mathrm{ad}}P)]$$

$$+\frac{g^2}{32\pi^2\varepsilon}\int \mathrm{d}^dx\{\mathrm{Tr}[C(-\tfrac{19}{6}\mathscr{D}^2\delta_{\beta\alpha}+\tfrac{11}{3}\mathscr{D}_\beta\mathscr{D}_\alpha-\tfrac{1}{3}F_{\beta\alpha})H^F_{\alpha\beta_{\mathrm{diag}}}]$$

$$+9\,\mathrm{Tr}(t_a^{\mathrm{ad}}Pt_a^{\mathrm{ad}}H^F_{\alpha\alpha_{\mathrm{diag}}})\},\tag{4.8}$$

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]^{\mathrm{pole}}_f=\frac{g^2}{(16\pi^2\varepsilon)^2}\int \mathrm{d}^dx[\tfrac{1}{12}\varepsilon(F_{\mu\nu}C^2F_{\mu\nu})-\tfrac{1}{2}\,\mathrm{Tr}(CP^2)$$

$$+\tfrac{1}{2}(1+\tfrac{1}{4}\varepsilon)g^2C_{ab}\varphi^{\mathrm{T}}T_aT_bD^2\varphi+2(1+\tfrac{1}{4}\varepsilon)\,\mathrm{Tr}(t_a^{\mathrm{ad}}Pt_a^{\mathrm{ad}}P)]$$

$$+\frac{g^g2}{32\pi^2\varepsilon}\int \mathrm{d}^dx\{\mathrm{Tr}[(-\tfrac{1}{6}\mathscr{D}^2\delta_{\beta\alpha}-\tfrac{1}{3}\mathscr{D}_\beta\mathscr{D}_\alpha-\tfrac{1}{3}F^{\mathrm{ad}}_{\beta\alpha})H^F_{\alpha\beta_{\mathrm{diag}}}$$

$$-\mathrm{Tr}(t_a^{\mathrm{ad}}Pt_a^{\mathrm{ad}}H^F_{\alpha\alpha_{\mathrm{diag}}})+2\mathrm{Tr}(\mathscr{D}^2H^g_{\mathrm{diag}})]\}.\tag{4.9}$$

The amplitude for 3(*g*) is

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]_g=\tfrac{1}{2}g^2\int \mathrm{d}^dx\;\mathrm{d}^dy\;G^F_{\mu\nu,ab}\{\mathrm{Tr}[T_a(G^F\tilde{D}_\nu)T_b(D_\mu G^F)^{\mathrm{T}}]$$

$$-\mathrm{Tr}[T_a(D_\mu G^F\tilde{D}_\nu)T_b(G^F)^{\mathrm{T}}]\}\tag{4.10}$$

and in the course of extracting the pole terms from (4.9) we need to adapt some results from (A7) in I in the light of the easily proved identities

$$(F^{\mathrm{ad}}_{\mu\nu}F^{\mathrm{ad}}_{\mu\nu})_{ab}\,\mathrm{Tr}(T_aT_b)=(F_{\mu\nu}RCF_{\mu\nu}),\tag{4.10a}$$

$$(F^{\mathrm{ad}}_{\mu\nu})_{ab}\,\mathrm{Tr}(F^{\varphi}_{\mu\nu}T_bT_a)=-\tfrac{1}{2}(F_{\mu\nu}RCF_{\mu\nu}),\tag{4.10b}$$

$$\mathrm{Tr}(F^{\varphi}_{\mu\nu}T_aF^{\varphi}_{\mu\nu}T_a)=\mathrm{Tr}(T^2F^{\varphi}_{\mu\nu}F^{\varphi}_{\mu\nu})-\tfrac{1}{2}(F_{\mu\nu}RCF_{\mu\nu}),\tag{4.10c}$$

obtaining

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]^{\mathrm{pole}}_g$$

$$=\frac{1}{2}\frac{g^2}{(16\pi^2\varepsilon)^2}\int \mathrm{d}^dx\{-\tfrac{2}{3}\varepsilon\;\mathrm{Tr}(T^2F^{\varphi}_{\mu\nu}F^{\varphi}_{\mu\nu})+\tfrac{5}{12}\varepsilon(F_{\mu\nu}RCF_{\mu\nu})$$

$$-2(1+\varepsilon)[\mathrm{Tr}(U''T_aU''T_a)-2g^2\varphi^{\mathrm{T}}T_aT_bU''T_bT_a\varphi]$$

$$+\tfrac{1}{2}(-17+\varepsilon)\,\mathrm{Tr}(CP^2)-9\,\mathrm{Tr}(t_a^{\mathrm{ad}}Pt_a^{\mathrm{ad}}P)-4[\mathrm{Tr}(U''^2T^2)-2g^2U_i(T^2T^2\varphi)_i]$$

$$+(14+\varepsilon)g^2P_{ab}\varphi^{\mathrm{T}}T_aT_bT^2\varphi-16g^2\varphi^{\mathrm{T}}T^2T^2D^2\varphi$$

$$-(10+\varepsilon)P_{ab}\,\mathrm{Tr}(U''T_aT_b)-\mathrm{Tr}(RP^2)$$

$$-(1+\tfrac{1}{4}\varepsilon)g^2R_{ab}\varphi^{\mathrm{T}}T_aT_bD^2\varphi-4(1+\tfrac{1}{8}\varepsilon)g^2C_{ab}\varphi^{\mathrm{T}}T_aT_bD^2\varphi\}$$

$$+\frac{g^2}{16\pi^2\varepsilon}\int \mathrm{d}^dx[2\,\mathrm{Tr}(T^2D^2H^F_{\mathrm{diag}})-\tfrac{1}{6}\,\mathrm{Tr}(R\,\Delta_{\beta\alpha}H^F_{\alpha\beta_{\mathrm{diag}}})$$

$$+\mathrm{Tr}(U''T_aT_b)H^F_{\alpha\alpha,ab_{\mathrm{diag}}}-g^2\varphi^{\mathrm{T}}T_aT_bT^2\varphi H^F_{\alpha\alpha,ab_{\mathrm{diag}}}+\mathrm{Tr}(t_a^{\mathrm{ad}}Pt_a^{\mathrm{ad}}H^F_{\alpha\alpha_{\mathrm{diag}}})$$
$$+\mathrm{Tr}(CPH^F_{\alpha\alpha_{\mathrm{diag}}})+\mathrm{Tr}(T_aU''T_aH^F_{\mathrm{diag}})-g^2\varphi^{\mathrm{T}}T_aT_bH^F_{\mathrm{diag}}T_bT_a\varphi$$
$$+P_{ab}\,\mathrm{Tr}(T_aT_bH^F_{\mathrm{diag}})]. \tag{4.11}$$

The next three diagrams, in figures 3(h, i, j), contain two dimension-three vertices and hence the products of propagators are two orders less singular than 3(e, f, g), and thus $N=2$ here. The full amplitudes are

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]_h=\tfrac{1}{2}g^4\int \mathrm{d}^dx\;\mathrm{d}^dy\,(G^F_{\mu\nu,ad}G^F_{\mu\nu,bc}+G^F_{\mu\nu,ac}G^F_{\mu\nu,bd})\varphi^{\mathrm{T}}T_bT_aG^FT_cT_d\varphi, \tag{4.12}$$

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]_i=\tfrac{1}{12}\int \mathrm{d}^dx\;\mathrm{d}^dy\;U'''_{ijk}G^F_{il}G^F_{jm}G^F_{kn}U'''_{lmn}, \tag{4.13}$$

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]_j=-\tfrac{1}{2}g^4\int \mathrm{d}^dx\;\mathrm{d}^dy\,(\varphi^{\mathrm{T}}T_aT_cG^FT_bT_d\varphi)G^g_{ab}G^g_{cd}, \tag{4.14}$$

giving rise to pole terms

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]^{\mathrm{pole}}_h$$
$$=\frac{g^2}{(16\pi^2\varepsilon)^2}\int \mathrm{d}^dx[(4+\varepsilon)(-4g^2P_{ab}\varphi^{\mathrm{T}}T_aT_bT^2\varphi+\mathrm{Tr}(CP^2)$$
$$-2g^2\varphi^{\mathrm{T}}T_aT_bU''T_bT_a\varphi-\tfrac{1}{2}g^2C_{ab}\varphi^{\mathrm{T}}T_aU''T_b\varphi)+\tfrac{1}{2}g^2\varepsilon C_{ab}\varphi^{\mathrm{T}}T_aT_bD^2\varphi$$
$$+2\varepsilon g^2\varphi^{\mathrm{T}}T^2T^2D^2\varphi]+\frac{g^2}{16\pi^2\varepsilon}\int \mathrm{d}^dx[4g^2(2\varphi^{\mathrm{T}}T_aT_bH^F_{\mathrm{diag}}T_bT_a\varphi$$
$$+\tfrac{1}{2}C_{ab}\varphi^{\mathrm{T}}T_aH^F_{\mathrm{diag}}T_b\varphi)+4g^2\varphi^{\mathrm{T}}T_aT_bT^2\varphi H^F_{\alpha\alpha,ab_{\mathrm{diag}}}$$
$$-2\,\mathrm{Tr}(CPH^F_{\alpha\alpha_{\mathrm{diag}}})-\mathrm{Tr}(t_a^{\mathrm{ad}}Pt_a^{\mathrm{ad}}H^F_{\alpha\alpha_{\mathrm{diag}}})], \tag{4.15}$$

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]^{\mathrm{pole}}_i$$
$$=(16\pi^2\varepsilon)^{-2}\int \mathrm{d}^dx\{-\tfrac{1}{4}(2+\varepsilon)[U''''_{ijk}U''''_{ijn}U''_{kn}$$
$$-2\,\mathrm{Tr}(U''^2T^2)+2\,\mathrm{Tr}(U''T_aU''T_a)]$$
$$+\tfrac{1}{24}\varepsilon U''''_{ijkm}U''''_{ijkl}(D^2\varphi)_m\varphi_l\}+\frac{1}{16\pi^2\varepsilon}\,\frac{1}{2}\int \mathrm{d}^dx\;U'''_{ijk}U'''_{ijl}H^F_{kl_{\mathrm{diag}}}, \tag{4.16}$$

$$W^{(2)}_{1\mathrm{PI}}[A,\varphi]^{\mathrm{pole}}_j$$
$$=\frac{g^2}{(16\pi^2\varepsilon)^2}\int \mathrm{d}^dx[(1+\tfrac{1}{2}\varepsilon)(g^2\varphi^{\mathrm{T}}T_aT_bU''T_bT_a\varphi$$
$$+\tfrac{1}{2}g^2C_{ab}\varphi^{\mathrm{T}}T_aU''T_b\varphi-\tfrac{3}{4}\,\mathrm{Tr}(CP^2)+\tfrac{1}{2}\,\mathrm{Tr}(t_a^{\mathrm{ad}}Pt_a^{\mathrm{ad}}P)$$
$$+2g^2P_{ab}\varphi^{\mathrm{T}}T_aT_bT^2\varphi)-\tfrac{1}{4}\varepsilon g^2(\varphi^{\mathrm{T}}T^2T^2D^2\varphi+\tfrac{1}{2}C_{ab}\varphi^{\mathrm{T}}T_aT_bD^2\varphi)]$$
$$-\frac{g^2}{16\pi^2\varepsilon}\int \mathrm{d}^dx[g^2\varphi^{\mathrm{T}}T_aT_bH^F_{\mathrm{diag}}T_bT_a\varphi+\tfrac{1}{2}g^2C_{ab}\varphi^{\mathrm{T}}T_aH^F_{\mathrm{diag}}T_b\varphi$$
$$+2g^2\varphi^{\mathrm{T}}T_aT_bT^2\varphi H^g_{ab_{\mathrm{diag}}}-\mathrm{Tr}(CPH^g_{\mathrm{diag}})] \tag{4.17}$$

where in deriving (4.16) we have made use of (A2.11) and (A2.16). We now come to four diagrams which contain one dimension-three and one dimension-four vertex and also one mixed vector–scalar line, and hence potentially have $N = 1$. The complete contributions are as follows:

$$W^{(2)}_{1\text{PI}}[A, \varphi]_k = g^3 \int \mathrm{d}^d x\, \mathrm{d}^d y \{G^F_{\mu\nu,ab}[-(D_\mu G^F_{\nu,c})^\mathrm{T} T_a G^F \{T_b, T_c\}\varphi + (G^F_{\nu,c})^\mathrm{T} T_a D_\mu G^F \{T_b, T_c\}\varphi], \tag{4.18}$$

$$W^{(2)}_{1\text{PI}}[A, \varphi]_l = g^2 \int \mathrm{d}^d x\, \mathrm{d}^d y\, c_{bcd} (\bar{G}^F_{\mu,c} T_a T_e \varphi)(\mathscr{D}_\mu G^g)_{ba} G^g_{de}, \tag{4.19}$$

$$W^{(2)}_{1\text{PI}}[A, \varphi]_m = -g^3 \int \mathrm{d}^d x\, \mathrm{d}^d y\, c_{cde}[(\bar{G}^F_{\gamma,ej} G^F_{\beta\alpha,db} + \bar{G}^F_{\beta,dj} G^F_{\gamma\alpha,eb})(\mathscr{D}_\beta G^F_{\gamma\alpha})_{ca} + (\mathscr{D}_\beta \bar{G}^F_\gamma)_{c,j} G^F_{\alpha\gamma,eb} G^F_{\alpha\beta,da}](\{T_a, T_b\}\varphi)_j, \tag{4.20}$$

$$W^{(2)}_{1\text{PI}}[A, \varphi]_n = g \int \mathrm{d}^d x\, \mathrm{d}^d y\, U'''_{lmn} G^F_{\mu,la} (G^F T_a (G^F \bar{D}_\mu)^\mathrm{T})_{mn}. \tag{4.21}$$

The respective pole contributions are

$$W^{(2)}_{1\text{PI}}[A, \varphi]^{\text{pole}}_k = \frac{3(1+\frac{1}{4}\varepsilon)g^2}{(16\pi^2\varepsilon)^2} \int \mathrm{d}^d x (2\varphi^\mathrm{T} T^2 T^2 D^2 \varphi + \tfrac{1}{2} C_{ab} \varphi^\mathrm{T} T_a T_b D^2 \varphi) + \frac{6g^3}{16\pi^2\varepsilon} \int \mathrm{d}^d x [\varphi^\mathrm{T} T^2 T_a (D_\mu H_{\mu,a})_{\text{diag}} + \tfrac{1}{4} C_{ab} \varphi^\mathrm{T} T_a (D_\mu H_{\mu,b})_{\text{diag}}], \tag{4.22}$$

$$W^{(2)}_{1\text{PI}}[A, \varphi]^{\text{pole}}_l = -\frac{\frac{1}{2}(1+\frac{1}{4}\varepsilon)g^4}{(16\pi^2\varepsilon)^2} \int \mathrm{d}^d x\, C_{ab} \varphi^\mathrm{T} T_a T_b D^2 \varphi - \frac{1}{2} \frac{g^3}{16\pi^2\varepsilon} \int \mathrm{d}^d x\, C_{ab} \varphi^\mathrm{T} T_a (H^F_{\mu,b} \bar{\mathscr{D}}_\mu)_{\text{diag}}, \tag{4.23}$$

$$W^{(2)}_{1\text{PI}}[A, \varphi]^{\text{pole}}_m = W^{(2)}_{1\text{PI}}[A, \varphi]^{\text{pole}}_n = 0. \tag{4.24}$$

The final pole contributions from 1PI diagrams arise from those in figures $3(o, p)$ which, with two dimension-four vertices and two mixed vector–scalar lines, have $N = 0$. Their amplitudes are given by

$$W^{(2)}_{1\text{PI}}[A, \varphi]_o = \tfrac{1}{4} g^2 \int \mathrm{d}^d x\, \mathrm{d}^d y \{[\bar{G}^F_{\mu,a} \bar{D}_\nu T_b (D_\mu G G^F)^\mathrm{T} - \bar{G}^F_{\mu,a} T_b (D_\mu G^F \bar{D}_\nu)^\mathrm{T}] T_a G^F_{\nu,b} - [\bar{G}^F_{\mu,a} \bar{D}_\nu T_b (G^F)^\mathrm{T} - \bar{G}^F_{\mu,a} T_b (G^F \bar{D}_\nu)^\mathrm{T}] T_a D_\mu G^F_{\nu,b}\}, \tag{4.25}$$

$$W^{(2)}_{1\text{PI}}[A, \varphi]_p = -\tfrac{1}{2} g^2 \int \mathrm{d}^d x\, \mathrm{d}^d y\, c_{abc} \{G^F_{\beta\gamma,cd} [\mathscr{D}_\alpha \bar{G}^F_\beta T_d (\bar{G}^F_\beta \bar{D}_\gamma)^\mathrm{T} + \bar{G}^F_{\alpha,b} T_d ((\mathscr{D}_\alpha \bar{G}^F_\beta)_a \bar{D}_\gamma)^\mathrm{T}] + \bar{G}^F_{\beta,c} T_d [(\bar{G}^F_{\alpha,b} \bar{D}_\gamma)^\mathrm{T} (\mathscr{D}_\alpha G^F_{\beta\gamma})_{ad} + ((\mathscr{D}_\alpha \bar{G}^F_\beta)_a \bar{D}_\gamma)^\mathrm{T} G^F_{\alpha\gamma,bd}] - (\bar{G}^F_{\beta,c} \bar{D}_\gamma) T_d [G^F_{\alpha,b} (\mathscr{D}_\alpha G^F_{\beta\gamma})_{ad} + (\mathscr{D}_\alpha \bar{G}^F_\beta)^\mathrm{T}_a G^F_{\alpha\gamma,bd}]\}, \tag{4.26}$$

which yield divergent terms

$$W^{(2)}_{1\text{PI}}[A,\varphi]^{\text{pole}}_{o} = -\frac{4(1+\frac{1}{8}\varepsilon)g^4}{(16\pi^2\varepsilon)^2}\int d^dx[\varphi^{\text{T}}T^2T^2D^2\varphi+\tfrac{1}{2}C_{ab}\varphi^{\text{T}}T_aT_bD^2\varphi] + \frac{2g^3}{16\pi^2\varepsilon}\int d^dx[(D_\mu\varphi)^{\text{T}}T^2T_aH^F_{\mu,a_{\text{diag}}}+\tfrac{1}{2}C_{ab}(D_\mu\varphi)^{\text{T}}T_aH^F_{\mu,b_{\text{diag}}}] \qquad (4.27)$$

and

$$W^{(2)}_{1\text{PI}}[A,\varphi]^{\text{pole}}_{p} = \frac{3(1+\frac{11}{12}\varepsilon)g^4}{(16\pi^2\varepsilon)^2}\int d^dx\, C_{ab}\varphi^{\text{T}}T_aT_bD^2\varphi - \frac{3}{2}\frac{g^3}{16\pi^2\varepsilon}\int d^dx\, C_{ab}D_\mu\varphi T_aH^F_{\mu,b_{\text{diag}}}. \qquad (4.28)$$

The products of propagators in the last four 1PI two-loop diagrams, figures 3(q–t), are less singular than $(x-y)^{-4}$ in four dimensions and hence give only finite contributions without any regularisation and produce no poles in ε:

$$W^{(2)}_{1\text{PI}}[A,\varphi]^{\text{pole}}_{q+\ldots+t} = 0. \qquad (4.29)$$

Finally we must include contributions from several counterterms deriving from the one-loop renormalisation in (2.22) and (2.32), represented in figures 3(u–x). We display the full amplitudes, which except for 3(v) have no finite part, for each type of counterterm, and then after inserting the values of $Z^{(1)}_A$, $Z^{(1)}_\varphi$, $V^{(1)}(\varphi)$, $(Z^{1/2}_v)^{(1)}$ and $(Z^{1/2}_f)^{(1)}$ given in (3.8), (3.13) and (3.14), we separate the divergences into their local and non-local pieces using the methods of I together with appendix 1.

We deal first with the terms associated with the matrix E in (2.32). These yield

$$W^{(2)}_{1\text{PI}}[A,\varphi]_u = \int d^dx\{-\text{Tr}[(Z_v^{-1/2})^{(1)}(\mathscr{D}^2G^g)_{\text{diag}}]-g^2\varphi^{\text{T}}T_aT_b(Z_f^{-1/2})^{(1)}\varphi G^g_{ab_{\text{diag}}}\}$$
$$= \frac{2g^2}{(16\pi^2\varepsilon)^2}\int d^dx\{2(1-\tfrac{1}{4}\varepsilon)\,\text{Tr}(CP^2)-2g^2P_{ab}\varphi^{\text{T}}T_aT_bT^2\varphi+\tfrac{1}{12}(F_{\mu\nu}C^2F_{\mu\nu})\} + \frac{2g^2}{16\pi^2\varepsilon}\int d^dx[-\text{Tr}(C\mathscr{D}^2H^g_{\text{diag}})+g^2\varphi^{\text{T}}T_aT_bT^2\varphi H^g_{ab_{\text{diag}}}]. \qquad (4.31)$$

Next, we consider counterterms quadratic in v appearing in (2.22). The amplitude is given by

$$W^{(2)}_{1\text{PI}}[A,\varphi]_v = \tfrac{1}{2}\int d^dx\{-\text{Tr}[(Z_AZ_v)^{(1)}(\Delta_{\alpha\beta}G^F_{\beta\alpha})_{\text{diag}}] - \text{Tr}[Z^{(1)}_vPG^F_{\alpha\alpha_{\text{diag}}}]+\varphi^{\text{T}}T_aT_bZ^{(1)}_\varphi\varphi G^F_{\alpha\alpha,ab_{\text{diag}}}\} \qquad (4.32)$$

and neglecting pieces which are regular as $\varepsilon\to 0$, this becomes

$$W^{(2)}_{1\text{PI}}[A,\varphi]^{\text{pole}}_v = \frac{g^2}{(16\pi^2\varepsilon)^2}\int d^dx\{(-\tfrac{5}{3}C_{ab}+\tfrac{1}{6}R_{ab})(-\tfrac{7}{4}\varepsilon C_{cb}F^a_{\mu\omega}F^c_{\mu\nu}+(6-\tfrac{7}{4}\varepsilon)P_{ac}P_{bc} + (6-2\varepsilon)g^2\varphi^{\text{T}}T_aT_bD^2\varphi)-4(4-\varepsilon)\,\text{Tr}(CP^2)+8(4-\varepsilon)g^2P_{ab}\varphi^{\text{T}}T_aT_bT^2\varphi\}$$

$$+\frac{g^2}{16\pi^2\varepsilon}\int d^dx\{\mathrm{Tr}[(-\tfrac{5}{3}C+\tfrac{1}{6}R)\Delta_{\beta\alpha}H^F_{\alpha\beta_{\mathrm{diag}}}]+2\,\mathrm{Tr}(CPH^F_{\alpha\alpha_{\mathrm{diag}}})$$
$$-4g^2\varphi^{\mathrm{T}}T_aT_bT^2\varphi H^F_{\alpha\alpha_{\mathrm{diag}}}\}. \qquad (4.33)$$

The counterterms with one v, one f lead to a contribution

$$W^{(2)}_{\mathrm{1PI}}[A,\varphi]_w=-g\int d^dx\{(D_\mu\varphi)^{\mathrm{T}}(Z_\varphi Z_f^{1/2})^{(1)}T_aG^F_{\mu,a_{\mathrm{diag}}}$$
$$-\varphi^{\mathrm{T}}(Z_\varphi Z_f^{1/2})^{(1)}T_a(D_\mu G^F_{\mu,a})_{\mathrm{diag}}$$
$$+(Z_v^{1/2})^{(1)}_{ab}[(D_\mu\varphi)^{\mathrm{T}}T_aG^F_{\mu,b_{\mathrm{diag}}}-\varphi^{\mathrm{T}}T_a(D_\mu G^F_{\mu,b})_{\mathrm{diag}}]\} \qquad (4.34)$$
$$=-\frac{12g^4}{(16\pi^2\varepsilon)^2}\int d^dx[3\varphi^{\mathrm{T}}T^2T^2D^2\varphi+C_{ab}\varphi^{\mathrm{T}}T_aT_bD^2\varphi]$$
$$+\frac{2g^3}{16\pi^2\varepsilon}\int d^dx\{3[(D_\mu\varphi)^{\mathrm{T}}T^2T_aH^F_{\mu,a_{\mathrm{diag}}}-\varphi^{\mathrm{T}}T^2T_a(D_\mu H^F_{\mu,a})_{\mathrm{diag}}]$$
$$+C_{ab}[(D_\mu\varphi)^{\mathrm{T}}T_aH^F_{\mu,b_{\mathrm{diag}}}-\varphi^{\mathrm{T}}T_a(D_\mu H^F_{\mu,b})_{\mathrm{diag}}]\}. \qquad (4.35)$$

Finally, the counterterms quadratic in f give rise to an amplitude

$$W^{(2)}_{\mathrm{1PI}}[A,\varphi]_x=\tfrac{1}{2}\int d^dx\{\mathrm{Tr}[(Z_\varphi Z_f)^{(1)}D^2G^F_{\mathrm{diag}}]$$
$$-\mathrm{Tr}[U''\{(Z_f^{1/2})^{(1)},G^F_{\mathrm{diag}}\}]-\mathrm{Tr}[V^{(1)''}G^F_{\mathrm{diag}}]\}. \qquad (4.36)$$

Using (A2.7), we obtain

$$W^{(2)}_{\mathrm{1PI}}[A,\varphi]^{\mathrm{pole}}_x$$
$$=(16\pi^2\varepsilon)^{-2}\int d^dx[(4-\varepsilon)g^2\,\mathrm{Tr}(U''^2T^2)+2(-5+\varepsilon)g^4U'_i(T^2T^2\varphi)_i$$
$$-\tfrac{1}{6}\varepsilon g^2\,\mathrm{Tr}(F^\varphi_{\mu\nu}F^\varphi_{\mu n}T^2)+(2+\varepsilon)g^4P_{ab}\varphi^{\mathrm{T}}T_aT_bT^2\varphi+4(4-\varepsilon)g^4\varphi^{\mathrm{T}}T^2T^2D^2\varphi$$
$$-g^2U'''_{ijk}U''_{jk}(T^2\varphi)_i+U'''_{ijk}U'''_{ijl}U''_{kl}-g^4U'''_{ijk}(T^2\varphi)_i(T_a\varphi)_j(T_a\varphi)_k$$
$$+U''_{ij}U''_{kl}U''''_{ijkl}+g^2U''_{ij}U''''_{ijkl}(T_a\varphi)_k(T_a\varphi)_l+2g^2\,\mathrm{Tr}(U''T_aU''T_a)$$
$$-6g^2P_{ab}\,\mathrm{Tr}(U''T_aT_b)+12g^4\varphi^{\mathrm{T}}T_aT_bU''T_bT_a\varphi+3g^4C_{ab}U'_i(T_aT_b\varphi)_i]$$
$$+\frac{1}{16\pi^2\varepsilon}\int d^dx[-2g^2\,\mathrm{Tr}[T^2D^2H^F_{\mathrm{diag}}]+\tfrac{1}{2}g^2U'''_{ijk}(T^2\varphi)_kH^F_{ij_{\mathrm{diag}}}$$
$$-\tfrac{1}{2}g^2\,\mathrm{Tr}(U''\{T^2,H^F_{\mathrm{diag}}\})-\tfrac{1}{2}U''_{ij}U''''_{ijkl}H^F_{kl_{\mathrm{diag}}}-\tfrac{1}{2}U'''_{ijk}U'''_{ijl}H^F_{kl_{\mathrm{diag}}}$$
$$+3g^2P_{ab}\,\mathrm{Tr}(T_aT_bH^F_{\mathrm{diag}})-6g^4\phi^{\mathrm{T}}T_aT_bH^F_{\mathrm{diag}}T_bT_a\varphi$$
$$-\tfrac{3}{2}g^4C_{ab}\varphi^{\mathrm{T}}T_aH^F_{\mathrm{diag}}T_b\varphi]. \qquad (4.37)$$

On adding (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), (4.11), (4.15)–(4.17), (4.22), (4.23), (4.27), (4.28), (4.31), (4.33), (4.35) and (4.37), we obtain the total divergent contribution to the two-loop vacuum potential

$$
\begin{aligned}
W^{(2)}_{1\mathrm{PI}}[A,\varphi]^{\mathrm{pole}}_{a+\dots+x}
&= \frac{1}{(16\pi^2\varepsilon)^2}\int \mathrm{d}^d x\,[g^2[\tfrac{35}{12}\varepsilon(F_{\mu\nu}C^2F_{\mu\nu}) - \tfrac{1}{12}\varepsilon(F_{\mu\nu}RCF_{\mu\nu}) - \tfrac{1}{2}\varepsilon\,\mathrm{Tr}(T^2F^{\varphi}_{\mu\nu}F^{\varphi}_{\mu\nu})] \\
&\quad + g^4[(-11+\tfrac{149}{24}\varepsilon)C_{ab} + (\tfrac{1}{2}-\tfrac{11}{24}\varepsilon)R_{ab}]\varphi^{\mathrm{T}}T_aT_bD^2\varphi \\
&\quad - (10+\tfrac{5}{4}\varepsilon)g^4\varphi^{\mathrm{T}}T^2T^2D^2\varphi + g^4(-\tfrac{11}{2}+2\varepsilon)U'_i(T^2T^2\varphi)_i \\
&\quad + g^4(\tfrac{3}{2}-\tfrac{1}{4}\varepsilon)C_{ab}U'_i(T_aT_b\varphi)_i - \tfrac{27}{4}\varepsilon g^2\,\mathrm{Tr}(t^{\mathrm{ad}}_aPt^{\mathrm{ad}}_aP) \\
&\quad + g^2\,\mathrm{Tr}\{[(-11+\tfrac{149}{24}\varepsilon)C + (\tfrac{1}{2}-\tfrac{7}{12}\varepsilon)R]P^2\} + \tfrac{1}{2}g^4\varphi^{\mathrm{T}}T^2U''T^2\varphi \\
&\quad + g^4(15-\tfrac{15}{2}\varepsilon)P_{ab}\varphi^{\mathrm{T}}T_aT_bT^2\varphi + g^4(6+\tfrac{1}{2}\varepsilon)\varphi^{\mathrm{T}}T_aT_bU''T_bT_a\varphi \\
&\quad + g^2\,\mathrm{Tr}[(3-\tfrac{1}{2}\varepsilon)U''^2T^2 - \tfrac{3}{2}\varepsilon U''T_aU''T_a] + \tfrac{1}{2}U''_{ij}U''''_{ijkl}U''_{kl} \\
&\quad + \tfrac{1}{24}\varepsilon U''''_{ijkm}U''''_{ijkl}(D^2\varphi)_m\varphi_l - g^2U''_{ij}U'''_{ijk}(T^2\varphi)_k + \tfrac{1}{2}(1-\tfrac{1}{2}\varepsilon)U''_{ij}U'''_{ikl}U'''_{jkl} \\
&\quad - (3+\tfrac{5}{2}\varepsilon)g^2P_{ab}\,\mathrm{Tr}(U''T_aT_b)] + \frac{1}{16\pi^2\varepsilon}\int \mathrm{d}^d x\,\mathrm{Tr}[C(\tilde{\mathscr{D}}^*\tilde{\mathscr{D}}H^{g})_{\mathrm{diag}}]
\end{aligned} \tag{4.38}
$$

after using (A2.13) and the identity

$$
\int \mathrm{d}^d x\,(H^F_{\mu}\tilde{\mathscr{D}}_{\mu})_{a_{\mathrm{diag}}}\varphi^{\mathrm{T}} = \int \mathrm{d}^d x\,(D_\mu H^F_{\mu,a})_{\mathrm{diag}}\varphi^{\mathrm{T}} = -\tfrac{1}{2}\int \mathrm{d}^d x\,H^F_{\mu,a}(D_\mu\varphi)^{\mathrm{T}}. \tag{4.39}
$$

With the help of the relation, proved in I,

$$
(\Delta H_{\Delta})_{\mathrm{diag}} = (16\pi^2\varepsilon)^{-1}a^{\Delta}_{2_{\mathrm{diag}}} \tag{4.40}
$$

we can remove the non-local terms in (4.38), replacing the numerical coefficients of $(F_{\mu\nu}C^2F_{\mu\nu})$ and $\mathrm{Tr}(CP^2)$ by $\tfrac{17}{6}\varepsilon$ and $(-11+\tfrac{161}{24}\varepsilon)$ respectively. The pole terms in (4.38) can now be cancelled by taking in $S^{(2)}[A,\varphi]$

$$
(Z^{(2)}_A)_{ab} = \frac{1}{\varepsilon}\frac{g^4}{(16\pi^2)^2}[\tfrac{34}{3}(C^2)_{ab} - \tfrac{1}{3}(RC)_{ab} - 2\,\mathrm{Tr}(T^2T_aT_b)], \tag{4.41a}
$$

$$
\begin{aligned}
Z^{(2)}_{\varphi} &= \frac{g^4}{(16\pi^2\varepsilon)^2}\{(20+\tfrac{5}{2}\varepsilon)T^2T^2 + (22-\tfrac{149}{12}\varepsilon)C_{ab}T_aT_b - (1-\tfrac{11}{12}\varepsilon)R_{ab}T_aT_b\} \\
&\quad - \frac{1}{\varepsilon}\frac{1}{(16\pi^2)^2}\frac{1}{12}S,
\end{aligned} \tag{4.41b}
$$

where

$$
S_{ij} = U''''_{iklm}U''''_{jklm}, \tag{4.42}
$$

$$
\begin{aligned}
V^{(2)}(\varphi) &= (16\pi^2\varepsilon)^{-2}[\![\tfrac{1}{2}g^4(-11+4\varepsilon)U'_i(T^2T^2\varphi)_i + g^4(\tfrac{3}{2}-\tfrac{1}{4}\varepsilon)C_{ab}U'_i(T_aT_b\varphi)_i \\
&\quad - \tfrac{27}{4}\varepsilon g^2\,\mathrm{Tr}(t^{\mathrm{ad}}_aPt^{\mathrm{ad}}_aP) + g^2\,\mathrm{Tr}\{[(-11+\tfrac{161}{24}\varepsilon)C + (\tfrac{1}{2}-\tfrac{7}{12}\varepsilon)R]P^2\} \\
&\quad + \tfrac{1}{2}g^4\varphi^{\mathrm{T}}T^2U''T^2\varphi - g^2U''_{ij}U'''_{ijk}(T^2\varphi)_k + 15g^4(1-\tfrac{1}{2}\varepsilon)P_{ab}\varphi^{\mathrm{T}}T_aT_bT^2\varphi \\
&\quad + g^4(6+\tfrac{1}{2}\varepsilon)\varphi^{\mathrm{T}}T_aT_bU''T_bT_a\varphi + \tfrac{1}{2}U''_{ij}U''''_{ijkl}U''_{kl} \\
&\quad + g^2\,\mathrm{Tr}[(3-\tfrac{1}{2}\varepsilon)U''^2T^2 - \tfrac{3}{2}\varepsilon U''T_aU''T_a] + \tfrac{1}{2}(1-\tfrac{1}{2}\varepsilon)U''_{ij}U'''_{ikl}U'''_{jkl} \\
&\quad - (3+\tfrac{5}{2}\varepsilon)g^2P_{ab}\,\mathrm{Tr}(U''T_aT_b)]\!].
\end{aligned} \tag{4.43}
$$

This result is exactly in accord with that obtained by van Damme (1982) since there the results were given for $V_u^{(1)}(\varphi)$ and $V_u^{(2)}(\varphi)$ where

$$V_u(\varphi) = V(Z_\varphi^{-1/2}\varphi). \tag{4.44}$$

With minimal subtraction $V_u(\varphi)$, and also Z_A, are independent of the gauge-fixing procedure. Thus $V_u^{(1)}$, $V_u^{(2)}$, $Z_A^{(1)}$, $Z_A^{(2)}$ define a gauge-invariant set of counterterms. As shown by van Damme (1982), the results agree with conventional calculations in some special cases.

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Appendix 1.

We collect here some results which are necessary to determine the residues of the poles arising from products of the Green functions $\mathcal{G}^F$ and G^g. The reader is referred to I for a fuller exposition of the heat kernel methods used to derive them.

In general, given an elliptic differential operator on flat space, of the form (3.3), its Green function G_Δ has a short distance expansion

$$G_\Delta(x, y) = G_0(x-y)a_0^\Delta(x, y) + R_1(x-y)a_1^\Delta(x, y) + R_2(x-y)a_2^\Delta(x, y) + H_\Delta(x, y) \tag{A1.1}$$

where the singular functions G_0, R_1, R_2 are specified in I (with $\mu = 1$) and $H_\Delta(x, y)$ contains no poles in ε and is regular for $x \sim y$, even when two derivatives act on it. a_0^Δ, a_1^Δ and a_2^Δ are analytic functions of x and y obtained in the asymptotic expansion of the heat kernel.

After using appendix A in I to evaluate the poles in products of the functions G_0, R_1 and R_2, the residues of these poles involve coincidence limits as $y \to x$ of a_0^Δ, a_1^Δ and a_2^Δ for the operators Δ_F and $\tilde{\mathcal{D}}^*\mathcal{D}$. From the well known recurrence relations for the coefficients a_n^Δ, we find for Δ given by (3.3)

$$a_{0_{\text{diag}}}^\Delta = 1, \tag{A1.2a}$$

$$a_{1_{\text{diag}}}^\Delta = -Y, \tag{A1.2b}$$

$$2a_{2_{\text{diag}}}^\Delta = c^\Delta - \tfrac{1}{3}\mathcal{D}^2 Y, \tag{A1.2c}$$

$$(D_\mu a_1^\Delta)_{\text{diag}} = \tfrac{1}{6}\mathcal{D}_\nu G_{\mu\nu} - \tfrac{1}{2}\mathcal{D}_\mu Y, \tag{A1.2d}$$

with

$$\mathcal{D}_\mu Y = \partial_\mu Y + [X_\mu, Y]$$

and c^Δ as given in (3.4b).

Moreover, it then follows from I in conjunction with (A1.2b, d) that

$$G_{\Delta_{\text{diag}}} = -(1/16\pi^2)(2/\varepsilon)Y + H_{\Delta_{\text{diag}}}, \tag{A1.3a}$$

$$(D_\mu G_\Delta)_{\text{diag}} = (16\pi^2\varepsilon)^{-1}\{\tfrac{1}{3}\mathcal{D}_\nu G_{\mu\nu} - \mathcal{D}_\mu Y\} + (D_\mu H_\Delta)_{\text{diag}}. \tag{A1.3b}$$

The particular results we need are obtained by substituting in (A1.2) and (A1.3) the values of X^F_μ, Y^F and X^g_μ, Y^g given in (3.5) and (3.6).

Appendix 2.

The gauge invariance of $U(\varphi)$ can be used to derive some useful results. We have

$$U(\varphi) = U(G\varphi), \qquad G = \exp\omega^\varphi. \tag{A2.1}$$

By extracting coefficients of ω, ω^2, ω^4 from (A2.1), we obtain

$$U'_i(T_a\varphi)_i = 0, \tag{A2.2}$$

$$\varphi^{\mathrm{T}}T_aU''T_b\varphi = \tfrac{1}{2}U'_i(\{T_a, T_b\}\varphi)_i, \tag{A2.3}$$

$$\begin{aligned} &U''''_{ijkl}(T_a\varphi)_i(T_a\varphi)_j(T_b\varphi)_k(T_b\varphi)_l \\ &\quad + 4U'''_{ijk}(T_a\varphi)_i(T_b\varphi)_j(T_aT_b\varphi)_k + 2U'''_{ijk}(T_a\varphi)_i(T_a\varphi)_j(T^2\varphi)_k \\ &\quad - 4\varphi^{\mathrm{T}}T_aU''T_aT^2\varphi - \tfrac{1}{6}C_{ab}\varphi^{\mathrm{T}}T_aU''T_b\varphi + \tfrac{1}{6}C_{ab}U'_i(T_aT_b\varphi)_i \\ &\quad + 2\varphi^{\mathrm{T}}T_aT_bU''T_bT_a\varphi + \varphi^{\mathrm{T}}T^2U''T^2\varphi + U'_i(T^2T^2\varphi)_i = 0. \end{aligned} \tag{A2.4}$$

Further important relations are obtained by differentiating the above. By taking $\partial/\partial\varphi_k$ of (A2.2) we have

$$U''_{kj}(T_a\varphi)_j + U'_j(T_a)_{jk} = 0 \tag{A2.5}$$

from which the identity

$$\varphi^{\mathrm{T}}T_aT^2U''T_a\varphi = U'_i(T^2T^2\varphi)_i \tag{A2.6}$$

follows on multiplication by $(T_aT^2\varphi)_i$.

Differentiating (A2.3) with respect to φ_k, we obtain

$$\begin{aligned} &U'''_{ijk}(T_a\varphi)_i(T_b\varphi)_j + U''_{ij}(T_a)_{ik}(T_b\varphi)_j + U''_{ij}(T_a\varphi)_i(T_b)_{ik} \\ &\quad + \tfrac{1}{2}U''_{ik}(\{T_a, T_b\}\varphi)_i + \tfrac{1}{2}U'_i(\{T_a, T_b\})_{ik} = 0 \end{aligned} \tag{A2.7}$$

which yields, in conjunction with (A2.3) and (A2.6),

$$U'''_{ijk}(T_a\varphi)_i(T_b\varphi)_j(T_aT_b\varphi)_k = U'_i(T^2T^2\varphi)_i - \varphi^{\mathrm{T}}T_aT_bU''T_bT_a\varphi \tag{A2.8}$$

and the consequent simplification of (A2.4)

$$\begin{aligned} &U''''_{ijkl}(T_a\varphi)_i(T_a\varphi)_j(T_b\varphi)_k(T_b\varphi)_l + 2U'''_{ijk}(T_a\varphi)_i(T_a\varphi)_j(T^2\varphi)_k \\ &\quad + U'_i(T^2T^2\varphi)_i + \varphi^{\mathrm{T}}T^2U''T^2\varphi - 2\varphi^{\mathrm{T}}T_aT_bU''T_bT_a\varphi = 0. \end{aligned} \tag{A2.9}$$

Taking $\partial/\partial\varphi_k$ of (A2.5), we have

$$U'''_{ijk}(T_a\varphi)_j + U''_{jk}(T_a)_{ji} + U''_{ij}(T_a)_{jk} = 0 \tag{A2.10}$$

which yields, on squaring,

$$U'''_{ijk}U'''_{ijl}(T_a\varphi)_k(T_a\varphi)_l = -2\,\mathrm{Tr}(U''^2T^2) + 2\,\mathrm{Tr}(U''T_aU''T_a). \tag{A2.11}$$

Moreover, setting $b = a$ in (A2.8) and differentiating again leads, with the help of (A2.10), to the identity

$$U''''_{ijkl}(T_a\varphi)_i(T_a\varphi)_j + U'''_{ikl}(T^2\varphi)_i - (\{U'', T^2\})_{lk} + 2(T_aU''T_a)_{lk} = 0. \tag{A2.12}$$

Differentiating (A2.10) with respect to φ_l, we obtain

$$U''''_{ijkl}(T_a\varphi)_j + U'''_{ijk}(T_a)_{jl} + U'''_{ljk}(T_a)_{ji} + U'''_{ijl}(T_a)_{jk} = 0. \tag{A2.13}$$

Now if we define the appropriate covariant derivative $\tilde{\mathscr{D}}$ acting on U''' by

$$(\tilde{\mathscr{D}}_\mu U''')_{ik} = \partial_\mu U'''_{ijk} + (A^\varphi_\mu)_{il} U'''_{ljk} + (A^\varphi_\mu)_{jl} U'''_{ilk} + (A^\varphi_\mu)_{kl} U'''_{ijl} \tag{A2.14}$$

we obtain on combining with (A2.13)

$$(\tilde{\mathscr{D}}_\mu U''')_{ijk} = U''''_{ijkl}(D_\mu\varphi)_l \tag{A2.15}$$

and hence, if $U(\varphi)$ is quartic,

$$U'''_{ijk}\mathscr{D}^2 U'''_{ijk} = (D^2\varphi)^{\mathrm{T}} S\varphi \tag{A2.16}$$

where S is as defined in (4.42).

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